showed a large deflection collapse in the predicted area at a 100% load value.

From the radome analysis results, it is observed that a large discrepancy exists between the bifurcation buckling and collapse loads. The primary reason is that considerable distortion of geometry occurs at loads below the predicted bifurcation buckling.

The finite element technique yielded excellent results for a complicated shell problem which included orthotropic sandwich-plate construction. The method is straightforward in use and general in application.

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Unsteady Temperature Distribution in Volume Reflectors

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FEW authors have considered the development of unsteady temperature distributions in semitransparent scattering materials under the influence of arbitrary incident radiative flux with other than specified temperature boundary conditions. The present work considers such a problem for an approximate radiative transfer model which allows analytic solutions. Such solutions are useful in the analysis of volume-reflecting heat shields considered for planetary entry environments in which incident radiative heating is dominant.[1] The model represents the unsteady, one-dimensional, radiative heating of a weakly absorbing, semitransparent medium with specified conductive heat flux at one boundary. The radiation field is modeled using an approximation[1] to the radiative heat flux obtained by solution of the Kubelka-Munk differential equations which have been successfully employed for many years in the paint and paper industry.[2,3] The radiation field in this model is characterized by two fluxes, a transmitted flux, I_T, and a reflected flux, I_R. It is assumed that internal radiative emission may be neglected with respect to the internally transmitted fluxes. This is justified for moderate temperatures and low absorption coefficients. The latter condition, of course, is necessary for a material to be a good volume reflector.[1] The index of refraction of the medium enveloping the scattering centers is assumed to be close to unity, eliminating the need for consideration of the surface reflectance at the boundaries of the slab.

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The Kubelka-Munk differential equations are

$$dI_T/dy = -(s+k)I_T + sI_R \tag{1a}$$

$$dI_R/dy = (s+k)I_R - sI_T \tag{1b}$$

where s and k are the Kubelka-Munk scattering and absorption coefficients and y is the coordinate measured normal to the boundary. Defining the radiative heat flux as

$$q_R = \pi(I_T - I_R) \tag{2}$$

subtraction of Eq. (1b) from Eq. (1a) leads to the following expression for the divergence of the radiative heat flux

$$dq_R/dy = -k\pi(I_T + I_R) \tag{3}$$

The radiative heat flux divergence may be expressed approximately in terms of scattering and absorption coefficients and position by solving Eqs. (1) simultaneously for I_T and I_R under the condition $k = 0$ and substituting into Eq. (3). Defining the arbitrary diffuse, incident, radiative flux q_0, and the over-all reflectance, R, the boundary conditions for the solutions of Eqs. (1) may be formulated as

$$I_T(0) = q_0/\pi, \; I_R(0)/I_T(0) = R \tag{4}$$

The solutions of Eqs. (1) with $k = 0$ which satisfy Eqs. (4) are found to be

$$I_T = q_0/\pi[1 - sy(1-R)] \tag{5a}$$

$$I_R = q_0/\pi[R - sy(1-R)] \tag{5b}$$

leading to the radiative heat flux divergence expression‡

$$dq_R/dy = -kq_0[R + 1 - 2sy(1-R)] \tag{6}$$

The reflectance R of a slab of semitransparent material of thickness δ is known from reflectance solutions of the Kubelka-Munk differential equations[1] as

$$R = \frac{(1/R_\infty)(R_B - R_\infty) - R_\infty(R_B - 1/R_\infty)\exp[S(1/R_\infty - R_\infty)]}{(R_B - R_\infty) - (R_B - 1/R_\infty)\exp[S(1/R_\infty - R_\infty)]} \tag{7}$$

where

$$R_\infty = 1 + k/s - [k/s(k/s+2)]^{1/2} \tag{8}$$

Here R_∞ is the reflectance of an infinite thickness of the media, $S = s\delta$ is the scattering power of the media for the thickness δ, and R_B is the rear surface reflectance. Equations (6–8) thus determine the interaction of the radiation and temperature fields through the radiative flux divergence for assigned values of k, s, δ, R_B, and q_0.

The temperature field is governed by the energy equation

$$\rho C \frac{\partial T}{\partial t} = K \frac{\partial^2 T}{\partial y^2} + q_0 k[1 + R - 2sy(1-R)] \tag{9}$$

where the source term is given by the radiative flux divergence in Eq. (6). It is convenient to define the following dimensionless parameters:

$$\tau = \alpha t/\delta^2, \quad \eta = y/\delta, \quad \theta = T/T_s, \quad \Phi = kq_0\delta^2/KT_s$$

which transform Eq. (9) into

$$\partial\theta/\partial\tau = \partial^2\theta/\partial\eta^2 + \Phi[1 + R - 2S(1-R)\eta] \tag{10}$$

Equation (10), together with appropriate initial and boundary conditions, provides an initial-boundary value problem for the determination of $\theta(\eta, \tau)$.

We restrict our consideration to the class of solutions in which the initial temperature is constant and the rear surface is a perfect insulator which transmits all incident unreflected radiation. Specifying the constant, dimensionless temperature gradient at the surface $y = 0$ as A, the initial and boundary conditions are

$$\theta(\eta, 0) = \theta_1, \qquad 0 < \eta < 1 \tag{11}$$

$$\theta_\eta(0, \tau) = A, \qquad \theta_\eta(1, \tau) = 0, \qquad \tau > 0 \tag{12a, b}$$

‡ Note that Eqs. (2–5) here differ from the corresponding equations given in Ref. 1. In that reference, the factors of π are absent. This difference exists because here the symbols I_T and I_R are considered to be radiant intensities, whereas in Ref. 1, they were regarded as half-fluxes. It should be noted that the π factors drop out of the analysis at this point leading to the same radiative heat flux divergence and energy equations as those found in the reference.

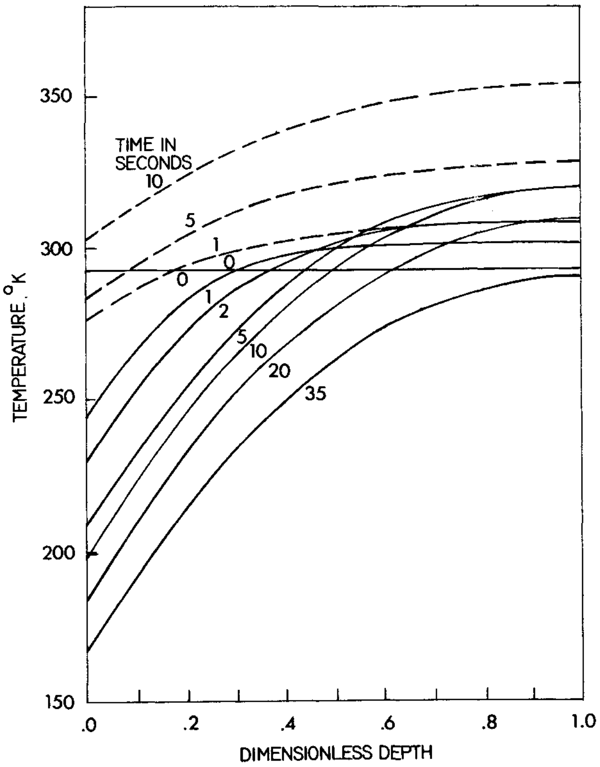

Fig. 1 Unsteady temperature histories in a scattering material. $q_0 = 1.02$ kw/cm^2, $\delta = 0.64$ cm, $k = 0.01$ cm^{-1}, $s = 10$ cm^{-1}, $K = 2.98 \times 10^{-2}$ w/cm-°K, $\alpha = 0.0132$ cm^2/sec, $R_B = 0.95$. Solid line—$A = 0.477$; dashed line—$A = 0.211$.

The problem formulated in Eqs. (10–12) may be solved by defining the transformation

$$\Theta = \theta + A[(\eta^2/2) - \eta] \tag{13}$$

which yields the following transformed problem with homogeneous boundary conditions:

$$\frac{\partial \Theta}{\partial \tau} = \frac{\partial^2 \Theta}{\partial \eta^2} - A + \Phi[1 + R - 2S\eta(1-R)] \tag{14}$$

$$\Theta(\eta, 0) = A\left(\frac{\eta^2}{2} - \eta\right) + \theta_1 \tag{15}$$

$$\Theta_\eta(0, \tau) = 0, \qquad \Theta_\eta(1, \tau) = 0 \tag{16a, b}$$

A solution of Eqs. (14–16) was obtained by expansion of Θ in the eigenfunctions of the related homogeneous problem[4] for which the eigenvalues are

$$\lambda_\eta = \eta^2 \pi^2, \qquad \eta = 0, 1, 2 \ldots \tag{17}$$

The solution of the original problem, using Eq. (13), is then given by

$$\theta = \theta_1 + \{\Phi[1 + R - S(1-R)] - A\}\tau + \Phi S(1-R)\frac{\eta^3}{3} - [\Phi S(1-R) + A]\frac{\eta^2}{2} + A\eta + \frac{1}{12}[\Phi S(1-R) - 4A] - \sum_{n=1}^{\infty}\left\{\frac{4S\Phi(1-R)}{\lambda_n^2}[1-(-1)^n] - \frac{2A}{\lambda_n}\right\}e^{-\lambda_n \tau}\cos(n\pi\eta) \tag{18}$$

The solution, Eq. (18), has been applied to estimate the unsteady temperature change from an initial temperature of 293°K in a 0.64 cm thick slab of a highly scattering, weakly absorbing material under an incident radiative flux of 1.02 kw/cm^2 and two different values of conductive boundary cooling flux. These solutions are displayed graphically in Fig. 1. For $A = 0.477$ it is seen that the temperature at the boundary drops rapidly while radiation absorption leads to a brief period of internal temperature rise followed by monotonic temperature decrease at all points in the slab. In contrast, for $A = 0.211$ the boundary temperature drops briefly and then joins the slab interior in a monotonic temperature rise governed by an excess of radiation absorption over surface conductive cooling.

Examination of Eq. (18) shows that the monotonic behavior at large values of the time is governed by the linear time-dependent terms. Positive values of the coefficient of τ lead to growth while negative values lead to decrease. The critical condition given by the vanishing of the coefficient of τ

$$A = \Phi[1 + R - S(1-R)] \tag{19}$$

corresponds to the exceptional case in which a steady state is attained. For the conditions of Fig. 1 a critical value of A of 0.422 is obtained. The resulting steady state and its transient development are shown in Fig. 2. For the assumed radiation model, the condition (19) may be shown to be deducible by integration of the radiation absorbed and a simple energy balance. It may also be shown that Eq. (18) reduces in the steady state to the solution given in Ref. 1 when Eq. (19) and the approximate reflectance relation used in Ref. 1 are applied.

Conclusions

It has been shown that a simple approximate model of the radiation field in a scattering, weakly absorbing medium may be employed to obtain reasonable analytic solutions for the unsteady temperature distribution in volume reflectors. Such a solution has been presented for the case of a constant incident radiative flux, constant radiative and thermal properties, and a specified boundary conductive flux. The solution demonstrates that asymptotically increasing or decreasing temperature solutions are obtained depending on the value of the conductive boundary flux. It is shown that the solution yields a critical value of boundary conductive flux for which a steady state is attained. The resulting critical condition was found to be identical to that resulting from a steady-state energy balance on the slab.

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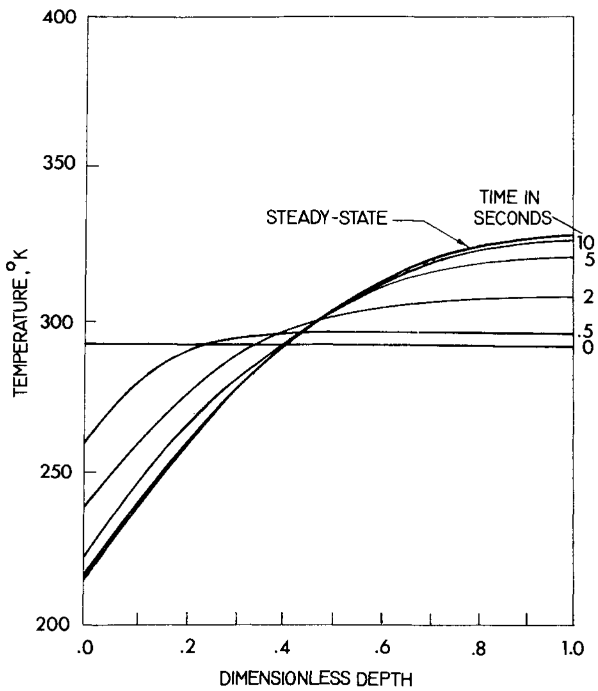

Fig. 2 Development of the steady state in a scattering material. $q_0 = 1.02$ kw/cm^2, $\delta = 0.64$ cm, $k = 0.01$ cm^{-1}, $s = 10$ cm^{-1}, $K = 2.98 \times 10^{-2}$ w/cm-°K, $\alpha = 0.0132$ cm^2/sec, $A = 0.422$, $R_B = 0.95$.

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An Approximate Distribution of Particle Mass Flux in a High Altitude Solid Propellant Rocket Plume

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Nomenclature

c	= constant in mass flux equation
F	= motor thrust
$g(r, \phi)$	= radial mass flux
$h(\phi), h(\xi)$	= angular dependent part of radial mass flux
I_{sp}	= motor specific impulse
k	= constant in fractional mass flow equation
R_e	= nozzle exit radius
r	= radial coordinate
$W(\phi)$	= particle mass flow within the conical angle ϕ
W_p	= total particle mass flow
$w(\phi), w(\xi)$	= fractional particle mass flow $[= W(\phi)/W_p]$
x_p	= mass fraction of metal oxide particle in exhaust
ϕ	= angular coordinate
ϕ_m	= limiting particle streamline angle
ξ	= nondimensional angular coordinate $(= \phi/\phi_m)$

I. Introduction

SOLID propellant rocket motors with thrusts from 100 to 10,000 lbf are used in space applications such as staging and orbit changes. When metalized propellants are used the exhaust products normally contain about 30% (by weight) metal oxide particles. Recurring problems with motors of this type are caused by the heating, pressures, and contamination on adjacent surfaces which are struck by the particles in the exhaust plume. Surface contamination is a particularly important problem because of the sensitivity to contamination of thermal control surfaces and solar cells. Because of this sensitivity, contamination by the particles remains a significant problem for separation distances far beyond those at which heating and pressures are no longer important.

Analyses which assess these effects must utilize some distribution of particle mass in the plume. Although there are computational codes[1,2] available which predict particle trajectories and densities in plumes, it would be useful to have some simple, but realistic and moderately accurate, analytical specification of the particle mass flux. The value of such a model is suggested by the widely used Hill-Draper[3] model for a gaseous vacuum plume. In fact, Hill and Draper's method of fitting an analytical approximation to detailed numerical gas dynamic plume solutions can also be applied to particle plumes. This Note presents a simple analytical model for the particle mass flux in a solid propellant motor plume which is derived in this manner.

II. Derivation

Detailed method-of-characteristics solutions for two-phase high-altitude plumes show that far from the nozzle exit the particle trajectories become straight and the particle plume resembles a spherical source flow. For these conditions the particle mass flow contained within a conical particle stream surface (Fig. 1) with half-angle ϕ is

$$W(\phi) = 2\pi \int_o^\phi r^2 g(r, \phi) \sin \phi \, d\phi \tag{1}$$

If ϕ_m is the limiting particle streamline which contains essentially all the particle mass flow, conservation of particle mass requires

$$W(\phi_m) = W_p = x_p(F/I_{sp}) \tag{2}$$

For a sourcelike flow the angular and radial dependence of the particle mass flux are separable:

$$g(r, \phi) = cW_p h(\phi)/r^2 \tag{3}$$

With this assumed form for the particle mass flux and the nondimensional variables $\xi = \phi/\phi_m$ and $w(\xi) = W(\phi)/W_p$, Eq. (1) becomes

$$w(\xi) = 2\pi c \phi_m \int_o^\xi h(\xi) \sin(\xi\phi_m) \, d\xi \tag{4}$$

Boundary values which must be imposed are $w(0) = 0$; $w(1) = 1$; $h(1) = 0$; and the constant c will be chosen so that $h(0) = 1$.

The quantity of primary interest is the angular dependent mass flux $h(\xi)$, which will be chosen to match detailed method-of-characteristics solutions. However, the procedure to be used

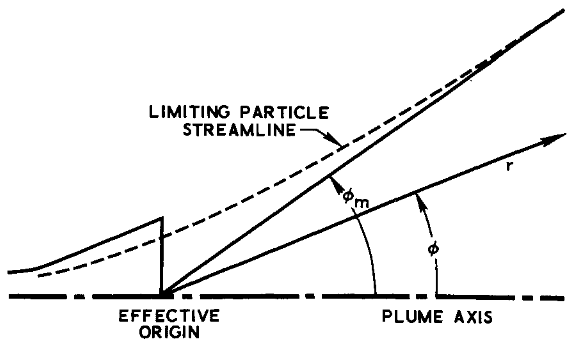

Fig. 1 Plume geometry and coordinate system.

TWO PHASE METHOD OF CHARACTERISTICS SOLUTIONS			
THRUST, lb_f	r/R_e	ϕ_m, deg	SYMBOL
2,000	26.8	35	O
2,000	80	35	Δ
70,000	13.5	40	◊
	$w(\xi) = \text{erf}(k\xi^2)$		———

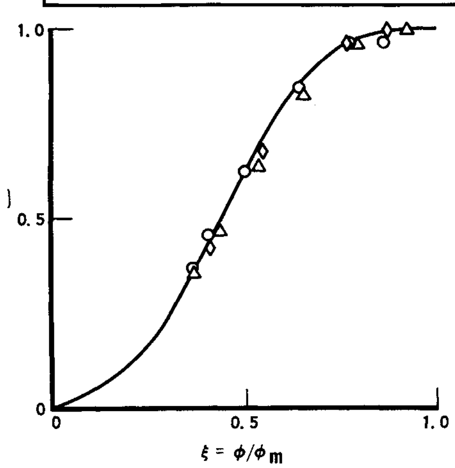

Fig. 2 Normalized particle fractional mass flow.

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